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NONLINEAR ANALYSIS AND OPTIMAL DESIGN OF DYNAMIC  
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DEPT OF MECHANICAL AND INDUSTRIAL EN.  
K D WILMERT ET AL. FEB 86 AFOSR-TR-86-2112 F/G 20/11

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method to handle various types of nonlinear constraints with the goal of reducing the number of analyses required to obtain an optimal design. To help overcome computational difficulties, DoD and NSF provided funds through two separate grants to purchase a research computer and other associated equipment as well as access to a supercomputer.



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**AFOSR-TR. 86-2112**

Annual Technical Report

February 1, 1985 through January 31, 1986

**NONLINEAR ANALYSIS AND OPTIMAL DESIGN OF DYNAMIC MECHANICAL  
SYSTEMS FOR SPACECRAFT APPLICATION**

Air Force Office of Scientific Research

Grant No. AFOSR-84-0076

Principal Investigators:

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February 1986

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### Summary

A nonlinear finite element analysis procedure already developed for planar mechanisms is being modified to handle complex mechanisms with sliding masses and mechanisms operating at relatively high speeds. Progress is also being made in developing a suitable nonlinear finite element analysis procedure for three-dimensional mechanisms. In both cases the analysis takes into account the effects of geometric and material nonlinearities, vibrational effects and coupling of deformations. In the optimal design area, a new algorithm has been developed for finding the minimum of a sum-of-squares objective function subject to general nonlinear constraints. The solution of preliminary examples indicate good results in terms of the total number of objective function evaluations to obtain an optimal design. Complete details of these investigations are included in the Appendix. To meet the extraordinary computational needs of this project, a separate research VAX 11/785 Computer and peripheral equipment were made available through a DoD research grant. The National Science Foundation also provided funds for some additional equipment as well as computational time on a supercomputer.

### Research Objectives

The objective of this research is to develop a nonlinear finite element dynamic analysis procedure for planar as well as spatial mechanisms that are frequently used in space structures. Included in the nonlinear analysis are the effects of curvature-displacement nonlinearity, nonlinearity due to extension or stretching (both caused by large deformations), material nonlinearity as well as combinations of these. In addition to the nonlinear analysis, an efficient optimal design method is to be developed to handle

objective functions composed of combinations of rigid body and deformation displacements involving geometric design variables as well as cross-sectional sizes of the members of the mechanisms subject to limitations on stresses and deformations. During the present reporting period it was proposed to:

- 1) Extend the nonlinear analysis procedure developed for two-dimensional mechanisms to three-dimensional mechanisms
- 2) Apply the optimization technique developed to simple mechanism problems

#### Significant Accomplishments

A major portion of the research during the reporting period was directed towards achieving these research objectives. Considerable progress has been made in both of these areas. A summary of significant accomplishments is presented below with complete details of the work included in the Appendix.

A non-linear finite element analysis procedure has already been developed for planar mechanisms to handle geometric and material nonlinearities. Complete details of the work are available in a paper entitled, "Finite Element Nonlinear Vibrational Analysis of Planar Mechanisms" by D. W. Tennant, K. D. Willmert and M. Sathyamoorthy presented at the ASME Winter Annual Meeting in Miami Beach, Florida in November 1985 and published in the Special Issue of Material Nonlinearity in Vibration Problems. The results of this investigation clearly indicate the need to include the nonlinearities in the dynamic analysis of mechanisms. There are, however, a number of difficulties in extending this approach to problems of mechanisms with sliding masses and mechanisms operating at relatively high speeds. To overcome these difficulties, the nonlinear analysis procedure is being modified to handle more complex planar mechanisms. Considerable progress has been made in modifying the already developed nonlinear analysis procedure and the required

computer programs for planar mechanism analysis. Numerical results for complicated cases of planar mechanisms, however, are still not available. It is expected that new results will be available for presentation at the AFOSR-SES (Society of Engineering Science) meeting to be held at the State University of New York at Buffalo in August 1986. Research is also in progress to develop a nonlinear finite element analysis procedure for three-dimensional mechanisms. The formulation of these problems includes the effects of material and geometric nonlinearities due to large deformations, and vibrational effects of members. A complete report on this will be sent to AFOSR as soon as results are available for some example problems.

The mathematical optimization techniques currently available for solving optimal design problems all require several iterations to obtain the best design. Some methods involve a large number of iterations, with each iteration requiring numerous analyses. These methods are adequate if the design problem is small, since computational times are relatively insignificant. However for large design problems, or ones in which a complex nonlinear analysis is required, it is extremely important that the optimization technique be very efficient, particularly in terms of the number of analyses required. In a paper entitled, "The Gauss Optimization Method For Problems With General Nonlinear Constraints" by T. E. Potter, K. D. Willmert, and M. Sathyamoorthy (presented at the 22nd Annual Meeting of the Society of Engineering Science held at the Pennsylvania State University, University Park, Pennsylvania, October 1985), a new algorithm is developed for finding the minimum of a sum-of-squares objective function subject to general nonlinear constraints. The solution of preliminary examples indicate good results in terms of the total number of objective function evaluations required by the algorithm to obtain an optimal design. The optimization techniques developed in this research, as extensions of the Gauss method to



handle various types of constraints, reduce the number of analyses required to obtain an optimal design thereby reducing the computational time significantly. This method is now being used to solve additional example problems.

Because of the very complex nonlinear analysis required, which must be repeated many times during the optimization process, a considerable amount of computer time is needed for this research. To meet these needs, a proposal entitled "Laboratory for Graphical Analysis of Nonlinear Deformations in Dynamic Structural-Mechanical Systems" was submitted to DoD under the DoD - University Instrumentation Program to purchase a separate research computer for this project. This resulted in a grant (No. AFOSR-85-0103) of \$101,567. Although the original proposal called for the purchase of a VAX 11/730, a very careful and thorough search for the best computer (with the available funds) resulted in the purchase of a much larger and faster VAX 11/785. Digital Equipment Corporation offered a sizable reduction in cost of its VAX 11/785 computer under the DEC Educational Discount Program. Because of this reduction and because of additional cost sharing by Clarkson University's School of Engineering, it was possible to purchase the VAX 11/785 at no additional cost to DoD.

Computer Equipment Purchased Through DoD-URIP Grant and Other Sources

I. VAX 11/785 COMPUTER HARDWARE

1. VAX 11/780 Packaged System Including:	\$102,750
(A) VAX 11/780 CPU	
(B) 2-Mbytes ECC MOS (64-K chip) Memory	
(C) H9652 UNIBUS Expansion Cabinet with BA11-K and DD11-DK	
(D) VAX/VMS License and Warranty	
2. TU80 9 Track Streaming Tape Drive with Cabinet	8,800
3. RUA81 456 Mbyte Fixed Disk	19,600
4. 2-Mbytes of Additional Memory	8,100
5. FP780 Floating Point Accelerator	8,960
6. Two DMF32-LP Communication Interfaces	5,250
7. 780 to 785 Upgrade Kit	80,000
8. 25 ft. RS232 Sync Cable	95
9. Two 300/1200/2400 Baud Telephone Modems	1,060
10. Installation	N/C
11. Insurance and Transportation	1,817
12. Miscellaneous - Installation of Power, Phone Lines, Terminal Lines, etc.	<u>1,756</u>
Total Computer Hardware Cost	\$238,188

II. COMPUTER DISPLAY TERMINAL

1. Tektronix M4115B Computer Display Terminal	\$19,950
2. Option N1: Warranty-Plus	1,025
3. Option 2B: Additional 512 Kbytes RAM	4,600
4. Option 23: Additional Four Planes of Display Memory	6,000
5. Option 09: 4695 Color Copier Interface	500
6. Option 42: Single Flexible Disk	1,700

7. Display Stand	750
8. Software Package	1,000
9. 4695 Color Graphic Copier	1,595
10. Option 42: Warranty-Plus	430
11. 4926 10 Mbyte Hard Disk	4,200
12. Option N1: Warranty-Plus	210
13. Shipping	<u>371</u>
Total Display Terminal Cost	\$42,331
III. SOFTWARE FOR VAX 11/785 COMPUTER	
1. VMS Operating System	N/C
2. FORTRAN License	5,170
3. DECNET Communication Software	2,950
4. IGL Graphics Software	2,677
5. PSI Access Software	<u>1,850</u>
Total Software Cost	\$12,647
IV. TOTAL HARDWARE & SOFTWARE COSTS	\$293,166

The total value of the equipment and software is \$293,166. Discounts and contributions from Digital Equipment Corporation, Tektronix, Clarkson University's College of Engineering and the Department of Mechanical and Industrial Engineering total \$191,599. Thus, the total cost of the hardware and software to DoD remained at \$101,567 as originally proposed. It should be noted that the capabilities of the VAX 11/785 system, including the Tektronix 4115 graphic terminal, are enormous compared to the originally proposed VAX 11/730. The VAX 11/785 system is 5 times faster than the VAX 11/730, has 4 Mbytes of memory (compared to only 1 Mbyte of memory for VAX 11/730), 456 Mbyte of disk

space (compared to 121 Mbyte of disk space) a total of 16 terminal lines, and a 9 track streaming tape drive (no tape drive was included in the original VAX 11/730 system).

Hardware and software were also purchased to tie the VAX 11/785 into Clarkson's campus-wide computer network. The physical link is through the University's VAX 11/780, but this is tied to the other computers on campus, which is linked to other universities though BITNET. This tie in of the VAX 11/785 allows the users of this research computer access to many of the other facilities of the university, such as high speed printers, digital plotters, laser printers, etc. It also allows researchers with terminal connected to the other computers on campus to sign on to the VAX 11/785 as though they were directly connected.

The Tektronix 4115B computer display terminal, which is connected to the VAX 11/785 computer, has recently been expanded to improve its capabilities. Both a 3-dimensional wire frame and a shaded surface option have been added. These options allow the terminal to locally manipulate 3-dimensional objects, such as rotating them in 3-dimensional space, removing hidden lines, drawing shaded surfaces, etc. These expansions result in this terminal being equivalent to a Tektronix 4129 terminal, which is the most recent high-end terminal introduced by Tektronix. The total cost of these options was \$16,475, which was made possible through contributions from Gleason Foundation, Proctor and Gamble, Tektronix, Corning Glass Works as well as the University's School of Engineering.

A recent grant from the National Science Foundation (Grant No. DMC-8500627), with M. Sathyamoorthy and K.D. Willmert as principal investigators, has included funds totaling \$10,715 for the further expansion of the graphic facilities. A Tektronix 4692 color graphics copier has been purchased and a Tektronix 4107 low resolution graphic terminal is anticipated as

a result of this grant. These purchases should complement the high resolution Tektronix 4115B terminal obtained through the DoD-University Research Instrumentation Program. In addition to these equipment funds, this NSF grant provides, as part of its Cooperative Program on the use of supercomputers, twenty-five hours of CPU time on a Cray X-MP supercomputer.

A summary of funding sources including the 1984-85 DoD-URIP Grant to purchase the VAX 11/785 research computer and other associated equipment (including upgrades) is given below:

DoD - University Research Instrumentation Program	\$101,567
Digital Equipment Corporation Contribution	120,852
Tektronix Discount and Contribution	11,110
National Science Foundation	10,715
Clarkson University's College of Engineering Contribution	56,999
Department of Mechanical & Industrial Engineering	3,176
Clarkson's Educational Resource Center	3,300
Gleason Foundation	7,000
Proctor and Gamble	5,300
Corning Glass Works	<u>337</u>
TOTAL	\$320,356

As a result of contributions from all of these sources, the total value of the equipment within this laboratory exceeds \$300,000 for an investment of only slightly over \$100,000 from DoD.

## Publications

### Technical Journals, Meetings and Conferences

1. M. Sathyamoorthy and K.D. Willmert, "Nonlinear Analysis and Design of Flexible Mechanisms," Presented at the Third Forum on Large Space Structures, Texas A&M University, College Station, Texas, July 8-10, 1985.
2. D.R. Boston, K.D. Willmert and M. Sathyamoorthy, "The Development and Application of Gauss' Nonlinearly Constrained Optimization Method," Paper published in the Special Issue of Computer Methods in Applied Mechanics and Engineering, edited by E.J. Haug and J.E. Taylor, North Holland Publishing Company, 1985.
3. T.E. Potter, K.D. Willmert and M. Sathyamoorthy, "The Gauss Optimization Method for Problems with General Nonlinear Constraints," Proceedings of the Society of Engineering Science Meeting, The Pennsylvania State University, October 1985, p. 10.
4. D.W. Tennant, K.D. Willmert, and M. Sathyamoorthy, "Finite Element Nonlinear Vibrational Analysis of Planar Mechanisms," Paper published in the Special Issue of Material Nonlinearity in Vibration Problems, AMD Vol. 71, ASME, November 1985, pp. 79-89.
5. Edward Kear III, M. Sathyamoorthy, and K.D. Willmert, "Vibration Analysis of Mechanisms With Geometric and Material Nonlinearities," Paper to be presented at the Combined AFOSR-SES (Society of Engineering Science) Meeting, State University of New York at Buffalo, August 1986.

Professional Personnel

1. K.D. Willmert, Professor of Mechanical and Industrial Engineering, Project Supervisor.
2. M. Sathyamoorthy, Associate Professor of Mechanical and Industrial Engineering, Project Supervisor.
3. T.E. Potter, Currently a Ph.D. student in Mechanical and Industrial Engineering working on optimization methods applicable to large mechanism design problems.
4. E.B. Kear III, Currently a Ph.D. student in Mechanical and Industrial Engineering working on nonlinear analysis techniques for deformation analysis of two and three-dimensional mechanisms.
5. M. El Sawy, Currently a Ph.D. student (supported by Egyptian Government) in Mechanical and Industrial Engineering working on nonlinear analysis techniques for deformation of three-dimensional mechanisms.
6. B. Whispell, Currently an M.S. student (partially supported by Alcoa Foundation) in Mechanical and Industrial Engineering working on computer graphics for display of three-dimensional mechanisms undergoing deformations.

#### APPENDIX

Paper No. 1 - D.W. Tennant, K.D. Willmert, and M. Sathyamoorthy, "Finite Element Nonlinear Vibrational Analysis of Planar Mechanisms," Published in Material Nonlinearity in Vibration Problems, AMD - Vol. 71, ASME, November 1985, pp. 79-89.

Paper No. 2 - T.E. Potter, K.D. Willmert, and M. Sathyamoorthy, "The Gauss Optimization Method for Problems with General Nonlinear Constraints," Paper presented at the 22nd Annual Technical Meeting of the Society of Engineering Science, The Pennsylvania State University, October 1985, p. 10.



## FINITE ELEMENT NONLINEAR VIBRATIONAL ANALYSIS OF PLANAR MECHANISMS

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### ABSTRACT

A finite element approach is presented in this paper for the nonlinear vibrational analysis of planar mechanisms. The analysis takes into account the effects of material and geometric nonlinearities on the dynamic behavior. The geometric nonlinearities included in this study are due to stretching of the neutral axis and the curvature-displacement nonlinearity, both caused by large deformations. The material nonlinearity is due to a nonlinear stress-strain relationship of the Ramberg-Osgood type. The analysis presented here makes use of hermite polynomials which ensure compatibility of curvature between elements. Using a variable correlation table, a global system of nonlinear equations are derived in terms of the global unknowns and the kinematics of the mechanism. A harmonic series technique is then used to obtain the steady state solutions to this system of nonlinear equations. Numerical results are presented for an example mechanism and the effects of the nonlinearities are discussed.

### INTRODUCTION

The importance of flexibility of linkages on the performance of high speed minimum-mass mechanisms is well recognized. A considerable amount of research has been done in this area in the last two decades. While it is desirable to develop analytical and numerical procedures that enable the design of rigid link mechanisms and robots to perform a given function with specified reliability, it is also important to evaluate the effects of flexibility of elastic members on their performance. It is known that a mechanism designed for operation at low speeds may not perform satisfactorily at high speeds due to the effects of large inertia forces and resulting elastic deformations. Thus it becomes necessary to include in the dynamic analysis of mechanisms, not only the effect of the rigid body motion, but also the flexibility of the linkages.

Most of the previous investigations in the area of elastic analysis of mechanisms have been carried out within the framework of the linear theory [1-17]. However, Viscomi and Ayre [18] used a Galerkin-type nonlinear analysis procedure to study the vibrations of a slider-crank mechanism. A later work by Sadler and Sandor [19] used the lumped parameter approach to a nonlinear dynamic model of an elastic linkage. The mechanism analyzed in this paper was a general four-bar linkage and the analytical model included the response coupling associated with both the transmission of forces at the pin joints and the dependence of the undeformed motion of a link on the elastic motion of other links. A

finite element analysis, with the aid of the piecewise linear method of Martin, was used by Sevak and McLarnan [20] to carry out the nonlinear analysis of a mechanism. Further nonlinear work dealing with the vibrations of elastic mechanisms are reported in References [21-23]. In a recent investigation, Thompson and Sung [24] used a variational formulation for the nonlinear finite element analysis of planar mechanisms considering geometric nonlinearities. Some experimental results were also presented.

This paper is concerned with the nonlinear vibrational analysis of general planar mechanisms. A finite element method is used which includes the effects of both geometric and material nonlinearities. The geometric nonlinearities included in this study are due to stretching of the neutral axis with partially constrained ends and a general curvature-displacement relationship, both caused by large deformations. The material nonlinearity is of the Ramberg-Osgood type with three parameters to represent the nonlinear stress-strain relationship [25-27]. Additional effects considered are transverse shear and rotatory inertia and changes in cross-section due to realistically proportioned members. The governing nonlinear differential equations are derived for each element in terms of the axial and transverse deformations, rotations, curvatures, and shear deformation angles. These equations are then assembled with the aid of a variable correlation table and the resulting global system of equations is solved using an iterative technique based on a harmonic series solution procedure.

#### FINITE ELEMENT FORMULATION

A finite element method is presented below for the nonlinear analysis of a general closed looped mechanism. The mechanism can be composed of various combinations of simple four bar chains, frame elements, sliders moving on fixed references, or sliders moving on rotating links. Each link is divided into one or more elements with each element having the local coordinate system as shown in Figure 1. If a slider is present, the masses  $M_1$  and  $M_2$  are located at ends 1 and 2, as shown. The length of the element  $\Delta$  is constant except for links with sliders moving along them.

The displacement vector of any point (a) on the element's neutral axis is given by:

$$S = (X_1 \cos \gamma + Y_1 \sin \gamma + x + u)i + (Y_1 \cos \gamma - X_1 \sin \gamma + w)j \quad (1)$$

where  $X_1$  and  $Y_1$  are the coordinates of end 1 of the element given by the rigid body motion. The coordinate  $x$  is measured along the element's neutral axis from 1 to 2 and  $\gamma$  is the angle between the rigid body position and the X-axis. The axial and transverse displacements of point (a) from the rigid body position are given by  $u$  and  $w$ , respectively. This equation takes into account both the rigid body motion and the elastic displacements and defines the position of any point along the neutral axis.

Differentiating Equation 1 with respect to time yields the velocity of any point (a). The unit vectors  $i$  and  $j$  move with coordinate system and vary with time. The angular velocity of any differential line segment on the neutral axis of the element is given by:

$$\dot{\gamma}_t + \dot{w}_{,xt} \quad (2)$$

where  $\dot{\gamma}_t$  is the derivative of  $\gamma$  with respect to time and  $\dot{w}_{,xt}$  is the derivative of the transverse displacement with respect to the local coordinate  $x$  and time  $t$ .

The kinetic energy due to rotation of the element is given by:

$$K.E.R = \frac{1}{2} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \rho A_x \dot{S}_{,t}^2 + \rho I_2 (\dot{\gamma}_t + \dot{w}_{,xt})^2 dx dy + \frac{1}{2} M_1 \dot{S}_{,t}^2_{x=0} + \frac{1}{2} M_2 \dot{S}_{,t}^2_{x=\Delta} \quad (3)$$

where  $\rho$  is the mass density and  $A_x$  and  $I_2$  are the cross sectional area and moment of inertia of the element respectively. The term  $\rho I_2 (\dot{\gamma}_t + \dot{w}_{,xt})^2/2$  represents

the effect of rotatory inertia. The kinetic energy due to beam bending associated with transverse shear [28] is:

$$K.E.B = \frac{1}{2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{\Delta} \{\rho I_z \alpha, \dot{\alpha}\} dx dy \quad (4)$$

where  $\alpha$  is a measure of the transverse shear angle.  
The strain energy for the element is given by:

$$U = \int_{vol} \int_0^{\epsilon} \sigma d\epsilon dvol + \int_{vol} \frac{1}{2} \tau_{xy} \gamma_{xy} dvol \quad (5)$$

where  $\sigma$ ,  $\epsilon$ ,  $\tau_{xy}$ , and  $\gamma_{xy}$  are the normal stress, normal strain, shear stress and shear strain, respectively. For a nonlinear material of the Ramberg-Osgood type [25-27], the relationship between stress and strain is:

$$\sigma = A \epsilon - B \epsilon^m \quad (6)$$

where  $A$  corresponds to Young's modulus  $E$ , and  $B\epsilon^m$  represents the nonlinear term.  $A$ ,  $B$  and  $m$  are constants for the particular material being considered. The above relationship, Equation 6, is valid only for positive strains. If the strain is negative, the following expression is used:

$$\sigma = A \epsilon + B (-\epsilon)^m \quad \text{if } \epsilon < 0 \quad (7)$$

The change in sign of the nonlinear term results in the same overall effect on the stress-strain relationship as for positive strain, i.e. either hardening or softening depending on the values of  $B$  and  $m$ .

Using the Ramberg-Osgood relationship, the following expression for the strain energy is obtained for positive strain  $\epsilon \geq 0$ :

$$U = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{\Delta} \left\{ \frac{1}{2} A \epsilon^2 - \frac{1}{m+1} B \epsilon^{m+1} \right\} dx dy + \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{\Delta} \frac{1}{2} G_{xy} \gamma_{xy}^2 dx dy \quad (8)$$

where  $x$  is the axial coordinate,  $y$  is the transverse coordinate, and  $G_{xy}$  is the shear modulus. When  $\epsilon < 0$  the equation is:

$$U = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{\Delta} \left\{ \frac{1}{2} A \epsilon^2 - \frac{1}{m+1} B (-\epsilon)^{m+1} \right\} dx dy + \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{\Delta} \frac{1}{2} G_{xy} \gamma_{xy}^2 dx dy \quad (9)$$

The nonlinear expression for the curvature  $R$  of a planar static beam undergoing large deformations is:

$$\frac{1}{R} = \frac{w_{,xx}}{(1 + w_{,x}^2)^{3/2}} \quad (10)$$

Thus the strain is

$$\epsilon = -\frac{y}{R} = -\frac{y w_{,xx}}{(1 + w_{,x}^2)^{3/2}} \quad (11)$$

Combining the geometric nonlinearities due to stretching of the neutral axis and the curvature-displacement nonlinearity, results in the expressions for normal and shear strain:

$$\epsilon = u_{,x} + \frac{1}{2} D_s w_{,x}^2 + \frac{y \alpha_{,x}}{(1 + D_b w_{,x}^2)^{3/2}} \quad (12)$$

$$\gamma_{xy} = w_{,x} + \alpha \quad (13)$$

Substituting these expressions for strain into the strain energy Equations (8) and (9) produces, for  $\epsilon \geq 0$  or  $\epsilon < 0$ :

$$\begin{aligned} S.E. = & \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{\Delta} b \left\{ \frac{1}{2} A \left( u_{,x} + \frac{1}{2} D_s w_{,x}^2 + \frac{y \alpha_{,x}}{(1 + D_b w_{,x}^2)^{3/2}} \right)^2 \right. \\ & - \frac{D_k}{m+1} B \left( \pm u_{,x} \pm \frac{1}{2} D_s w_{,x}^2 + \frac{y \alpha_{,x}}{(1 + D_b w_{,x}^2)^{3/2}} \right)^{m+1} \\ & \left. + \frac{1}{2} G_{xy} (w_{,x} + \alpha)^2 \right\} dx dy \quad (14,15) \end{aligned}$$

For equation (15) negative sign is applicable in all the terms containing  $\pm$  signs and is valid for  $\epsilon < 0$ . In Equations (12), (14) and (15),  $D_k$ ,  $D_b$  and  $D_s$  are tracing constants representing the effects of material nonlinearity, geometric nonlinearity due to curvature, and geometric nonlinearity due to stretching of neutral axis, respectively. Each tracing constant is equated to unity when that particular nonlinearity is being considered and is equated to zero when it is not.

In order to represent realistically proportioned members, changes in cross section are included. Each element is divided into sections of varying lengths with constant area. The integrations involved in the element equations are carried out in a piecewise fashion with the area in each section taken as a constant. This procedure provides a reasonable approximation of variable cross sectional members without having to resort to large numbers of elements.

The Lagrangian function  $L$  is defined as

$$L = \sum_{k=1}^S \sum_{i=1}^{N_k} (K.E.R + K.E.B - S.E.)_{ik} \quad (16)$$

where  $N_k$  is the total number of elements in the  $k$ th link and  $S$  is the number of links in the mechanism. Substituting Equations (3), (4) and (14) into Equation (16), the Lagrangian  $L$  can be expressed in terms of the displacements  $u$  and  $w$ , the shear angle  $\alpha$ , and the rigid body motion.

Hermite polynomials are used to approximate  $u$ ,  $w$ , and  $\alpha$  in order to satisfy the boundary conditions of various types of mechanisms easily and to ensure interelement compatibility. The axial deformation  $u$  is approximated by a linear shape function given by

$$u(x,t) = U_1(t) N_1(x) + U_2(t) N_2(x) \quad (17)$$

Similarly, fifth degree polynomial shape functions are used to approximate the transverse deformation  $w$ :

$$\begin{aligned} w(x,t) = & W_1(t) H_{11}(x) + \bar{W}_1(t) H_{21}(x) + \bar{m}_1(t) H_{31}(x) \\ & + W_2(t) H_{12}(x) + \bar{W}_2(t) H_{22}(x) + \bar{m}_2(t) H_{32}(x) \quad (18) \end{aligned}$$

The shear angle  $\alpha$  is also approximated by a fifth degree polynomial in order to make it compatible with the transverse displacement  $w$ . Therefore  $\alpha$  is assumed to be:

$$\begin{aligned} \alpha(x,t) = & \bar{\alpha}_1(t) H_{11}(x) + \bar{\psi}_1(t) H_{21}(x) + \bar{\lambda}_1(t) H_{31}(x) \\ & + \bar{\alpha}_2(t) H_{12}(x) + \bar{\psi}_2(t) H_{22}(x) + \bar{\lambda}_2(t) H_{32}(x) \quad (19) \end{aligned}$$

where  $\psi$  and  $\lambda$  are the first and second derivatives of  $\alpha$ , respectively.  
The Hermite polynomials are given by:

$$N_1(x) = 1 - e \quad ; \quad N_2(x) = e \quad (20)$$

$$H_{11}(x) = 1 - 10e^3 + 15e^4 - 6e^5 \quad ; \quad H_{12}(x) = 10e^3 - 15e^4 + 6e^5$$

$$H_{21}(x) = \Delta(e - 6e^3 + 8e^4 - 3e^5) \quad ; \quad H_{22}(x) = \Delta(-4e^3 + 7e^4 - 3e^5)$$

$$H_{31}(x) = \Delta^2(e^2 - 3e^3 + 3e^4 - e^5)/2 \quad ; \quad H_{32}(x) = \Delta^2(e^3 - 2e^4 + e^5)/2 \quad (21)$$

where,  $e = x/\Delta$ .

A transformation of coordinates is now introduced to change from the moving coordinate system associated with the elements to global coordinates. Only  $U_1$ ,  $W_1$ ,  $U_2$  and  $W_2$  need to be transformed. The other coordinates are angles or derivatives of angles which are not directional on the X, Y coordinate system used. The transformations are:

$$\begin{aligned} U_1 &= \bar{U}_1 \cos \phi_1 - \bar{W}_1 \sin \phi_1 \quad ; \quad U_2 = \bar{U}_2 \cos \phi_2 - \bar{W}_2 \sin \phi_2 \\ W_1 &= \bar{U}_1 \sin \phi_1 + \bar{W}_1 \cos \phi_1 \quad ; \quad W_2 = \bar{U}_2 \sin \phi_2 + \bar{W}_2 \cos \phi_2 \end{aligned} \quad (22)$$

For pin connections the transformation angles  $\phi_1$  and  $\phi_2$  are set equal to  $-\gamma$  (the rigid body angle) which transforms the coordinates back to the global coordinates. For sliders moving on rotating links the transformation becomes more involved. In this case, the deformation of the driver link must be transformed to correspond to the axial and transverse deformation of the output link [14].

Substituting the expressions from Equation (22) into Equations (17) and (18), the global coordinates for the system are then:

$$\bar{q} = [\bar{U}_1 \bar{W}_1 \bar{\theta}_1 \bar{m}_1 \bar{\alpha}_1 \bar{\psi}_1 \bar{\lambda}_1 \bar{U}_2 \bar{W}_2 \bar{\theta}_2 \bar{m}_2 \bar{\alpha}_2 \bar{\psi}_2 \bar{\lambda}_2]^T \quad (23)$$

The Lagrangian function is then written in terms of the transformed element coordinates. Differentiating the Lagrangian with respect to the element coordinates, the following element equations are obtained:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_t} \right) - \frac{\partial L}{\partial q} = 0 \quad (24)$$

In differentiating the expressions for the kinetic and strain energies in Equations (3), (4) and (14), it must be remembered that  $\Delta$ , which is the upper limit of integration, is a function of time. The operations carried out in Equation (24) results in a system of nonlinear element differential equations. Assembling the element matrices for the particular mechanism being solved results in the global system of equations:

$$M \ddot{Q}_{,tt} + C \dot{Q}_{,t} + (K_e + K_n) \bar{Q} = \bar{F}(t) \quad (25)$$

The M, C,  $K_e$  and  $K_n$  matrices are all functions of time. The C matrix results from the kinetic energy of the system. No damping was included in the formulation of the problem. The  $C \dot{Q}_{,t}$  term was found to be small and thus was ignored in the analysis. The matrix  $K_e$  is the linear portion of the total stiffness matrix. It is a function of rigid body motion, but is not a function of the deformations  $\bar{Q}$ . The matrix  $K_n$ , however, is the nonlinear portion of the stiffness matrix. It is a function of the deformations  $\bar{Q}$ . Equation (25) is thus a nonlinear system of differential equations.

The derivation of the finite element Equation (25) is based on the assumption of positive strains  $\epsilon$ . If the strain is negative a similar derivation is possible, based on Equation (15) for the strain energy rather than Equation (14). The only difference in the resulting Equation (25) is in the stiffness matrix. Wherever an  $\epsilon^{-1}$  occurs in the original formulation, it becomes  $(-\epsilon)^{-1}$  for negative strains. All other negative signs resulting from the introduction of  $-\epsilon$  cancels out in the differentiations required. Thus in order to handle both

positive and negative strains the terms involving  $\epsilon^{m-1}$  in the stiffness matrix were replaced by  $|\epsilon|^{m-1}$ .

In order to solve the nonlinear system of Equation (25), an iterative approach was used. First the equations were solved using the linear terms only, i.e. the  $K_n$  matrix was ignored. This was accomplished by setting all of the tracing constants  $D_b$ ,  $D_s$  and  $D_k$  equal to zero. The solution  $\bar{Q}$  for the linear equations was then used to determine values for the nonlinear stiffness matrix  $K_n$ . Equation (25) was then solved again for new values of  $\bar{Q}$ , and the process repeated. Experience showed that this procedure converged in from 3 to 5 iterations. To solve Equation (25) for  $\bar{Q}(t)$ , for particular  $K_n$ , a harmonic series solution method was used similar to that of Bahgat and Willmert [14]. This approach overcomes problems with stability, due to the time varying nature of the matrices, that sometimes result from an eigenvalue technique. The steady state solution is obtained without adding artificial damping. The solution, without the  $C \bar{Q}_{,t}$  term, is given by:

$$\bar{Q}(t) = \sum_{n=0}^N (K - n^2 \omega^2 M)^{-1} (\bar{A}_n \cos n\omega t + \bar{B}_n \sin n\omega t) \quad (26)$$

where  $\omega$  is the input crank speed, and  $\bar{A}_n$  and  $\bar{B}_n$  are solutions to the linear equations:

$$\bar{F}(t_k) = \sum_{n=0}^N \bar{A}_n \cos n\omega t_k + \sum_{n=0}^{N-1} \bar{B}_n \sin n\omega t_k ; k = 0, 1, \dots, 2N-1 \quad (27)$$

where  $\bar{B}_0$  is set equal to zero. The values of  $t_k$  are the times at  $2N$  equal time increments per revolution of the input crank given by:

$$t_k = \frac{\pi k}{N\omega} \quad \text{for } k = 0, 1, \dots, 2N-1 \quad (28)$$

Computational experience indicates that a fairly accurate solution is obtained using only a few terms in Equation (26). As the number of terms increases the components of the matrix  $(K - n^2 \omega^2 M)$  grow and thus the inverse  $(K - n^2 \omega^2 M)^{-1}$  becomes small. The summation can therefore be truncated to reduce computational time.

The stress in the links is calculated by evaluating the strains from Equation (12). The stress can then be determined at any point in an element using Equations (6) and (7). To find the maximum stress in an element the maximum strain must be found. Setting the first derivative of Equation (12) to zero and solving the resulting expression, the position of the maximum strain is determined. Once the location is known, the maximum strain and stress can be evaluated.

The above formulation is based on the use of the shear angle  $\alpha$ , which is appropriate particularly for short members. For long slender links this quantity is not required. The elimination of  $\alpha$  reduces the size of the problem considerably since the nodal deformations  $\alpha_1$ ,  $\psi_1$ ,  $\lambda_1$ ,  $\alpha_2$ ,  $\psi_2$  and  $\lambda_2$  would no longer be present. For long slender members:

$$\alpha = -w_{,x} \quad (29)$$

Using this expression, the equation for strain energy (14) for positive shear  $\epsilon$  reduces to:

$$\begin{aligned} \text{S.E.} = & \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^A b \left( \frac{1}{2} A (u_{,x} + \frac{1}{2} D_s w_{,x}^2 - \frac{y w_{,xx}}{(1 + D_b w_{,x}^2)^{3/2}})^2 \right. \\ & \left. - \frac{D_k}{m+1} B (u_{,x} + \frac{1}{2} D_s w_{,x}^2 - \frac{y w_{,xx}}{(1 + D_b w_{,x}^2)^{3/2}})^{m+1} \right) dx dy \quad (30) \end{aligned}$$

A similar expression exists for negative strain. The kinetic energy also changes if  $\alpha$  is not present. The energy associated with transverse shear, Equation (4), is eliminated and thus Equation (3) represents the total kinetic energy of the

element. Using a procedure similar to the method outlined above, vibrational equations of the same form as Equation (25) can be obtained, but they will be smaller in size. However, nonlinear terms still exist in the stiffness matrix due to the material and geometric nonlinearities. The method of solution is thus identical to that outlined earlier.

#### EXAMPLE PROBLEMS

The following example is presented to illustrate the method of solution. The nonlinearities due to neutral axis stretching, and complex curvature-displacement and stress-strain relationships are all considered. A four-bar linkage, as shown in Figure 2, is used as the example with all of the members flexible and made of the same material. The data for the mechanism is: Length of input crank (AB) = 5.0 in, Length of coupler (BC) = 11.0 in, Length of rocker (CD) = 10.5 in, Fixed distance (AD) = 10.0 in, Cross section of links a rectangular, Height of rectangle = 1.0 in, and Width of rectangle = 0.25 in. The position of the input crank is zero degrees at  $t = 0$  and the direction of rotation is counterclockwise. The mechanism is divided into three elements with each link in the mechanism taken as an element. The boundary conditions are that only moment and shear terms exist for the input crank's driven end (A). For the pin connections between links, there are deformations, rotations and shear terms, and for the rocker's fixed point there are only rotation and shear terms.

First, the deformations in the mechanism were determined with the shear angle  $\alpha$  present. In this case the crank link was rotated at 100 rad/sec. The material properties, approximating aluminum, were as follows:  $A = 10.87 \times 10^6$  lb/in<sup>2</sup>,  $B = 0.8387 \times 10^{11}$  lb/in<sup>2</sup>,  $m = 3.0$ , and Mass density = 0.0002536 lb-sec<sup>2</sup>/in<sup>4</sup>. Three separate procedures were used to obtain numerical results. First the problem was solved using the linear analysis method of Bahgat and Willmert [14], with  $E = A$ . Next the method of this paper was used with the tracing constants equal to zero. Thus a linear analysis was obtained. Finally the method was applied with all tracing constants equal to one, i.e. a full nonlinear analysis. A representative deformation  $\bar{U}_1$  as a function of crank position is shown in Figure 3. This is the horizontal deformation of the free end of the crank link. As can be seen, the three curves are very similar. The effect of the shear angle  $\alpha$  is to increase the deformation slightly. For this slow speed the linear and nonlinear analyses were almost identical.

The same problem was solved again at a higher speed of 200 rad/sec. The resulting deformation  $\bar{U}_1$  is shown in Figure 4. As can be seen, high frequency oscillations started to appear, with greater separation between the three analyses. At even higher speed these oscillations became more predominant to the point of instabilities in the motion at very high speeds.

The revised form of the analysis equations was considered next, i.e. the form without the shear angle  $\alpha$ . Here a crank speed of 150 rad/sec was used. A comparison was made of the effects of the various nonlinearities on the deformations and stresses as compared to the linear analysis. Figures 5 and 6 show a comparison of the linear and nonlinear deformations  $\bar{U}_2$  (the horizontal deformation of the free end of the output link) caused separately by geometric and material nonlinearities. Figures 7 and 8 show the maximum stresses in the connecting link of the mechanism. As expected, the material nonlinearity of the Ramberg-Osgood type results in deformations which are greater in magnitude than those obtained using a linear elastic model. The maximum stress decreased due to the presence of the term  $B\epsilon^m$  subtracted from the linear stress expression.

The geometric nonlinearities considered, namely curvature displacement and stretching of the neutral axis, both due to large deformations produced mixed results with deformations reduced at some points and increased at other points. The effect of the geometric nonlinearities would be expected to produce a stiffening of the members [28] of the mechanism and thus produce smaller deformations. The increased deformations in this case might be due to the fact that the deformations are in relationship to the entire mechanism and not just to an individual beam element.

#### CONCLUSIONS

The nonlinear analysis procedure, using a finite element technique, is an

effective method of calculating the steady state deformations and stresses in a mechanism. Significant differences can occur between the linear and nonlinear approaches. This was particularly true for the stresses in the example considered in this work. Research is still needed on the overall effect of the shear angle  $\alpha$ , and a more complete picture of the nonlinear terms in the analysis would be of value. Additional nonlinear effect should also be investigated, such as the effect on the translations of one link due to large deformations of the other links.

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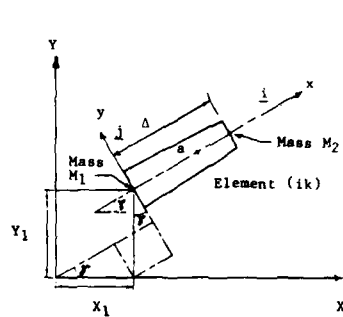


Fig. 1 General Element

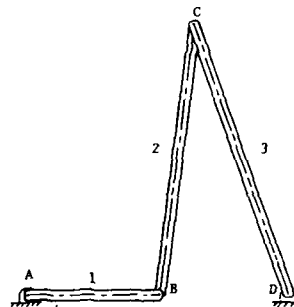


Fig. 2 Four Bar Mechanism

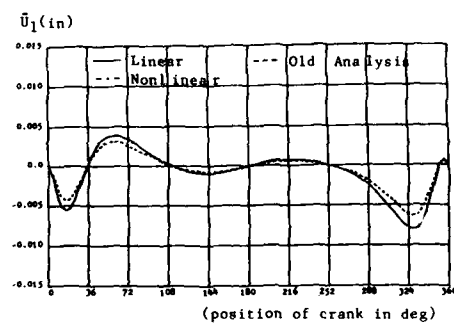


Fig. 3 Deformations  $\bar{U}_1$  for Aluminum at 100 rad/sec

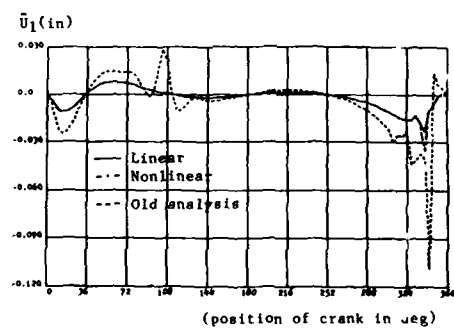


Fig. 4 Comparison of Deformations  $\bar{U}_1$  at 200 rad/sec

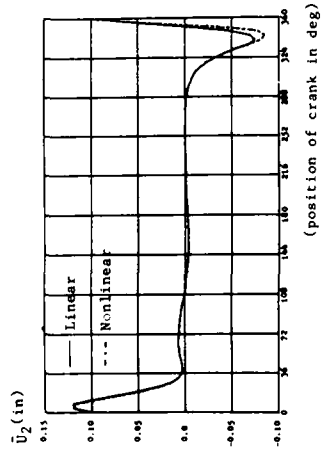


Fig. 5 Comparison of Deformations  $\bar{U}_2$  at 150 rad/sec for Geometric Nonlinearities

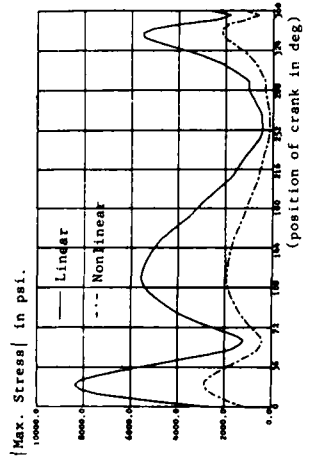


Fig. 7 Maximum Stresses in Element 2 at 150 rad/sec for Geometric Nonlinearities

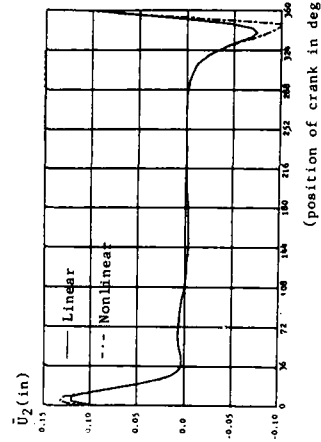


Fig. 6 Comparison of Deformations  $\bar{U}_2$  at 150 rad/sec for Material Nonlinearity

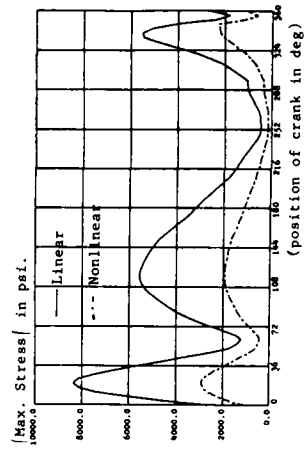


Fig. 8 Maximum Stresses in element 2 at 150 rad/sec for Material Nonlinearity

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ABSTRACT

A new algorithm is presented for finding the minimum of a nonnegative objective function subject to general nonlinear constraints. This algorithm, based on Gauss' method for unconstrained problems, is developed as an extension to the Gauss constrained technique for linear constraints. The derivation of the algorithm, using a Lagrange multiplier approach, is based on the Kuhn-Tucker conditions so that when the iteration process terminates these conditions are automatically satisfied. A feasible design is maintained throughout the iteration process. The solution of preliminary examples indicate excellent results in terms of the number of objective function evaluations required by the algorithm to obtain an optimal design.

INTRODUCTION

The optimal design of many complex structural and mechanical systems is hindered by the large computational times involved. Most currently available optimization techniques require a large number of analyses to obtain the optimal design. For small problems, or ones in which the analysis is simple, these methods are adequate; however, for large problems, or where a time consuming analysis is required, more efficient optimization methods are needed. The goal of this research was to develop such methods, particularly techniques applicable to mechanical mechanism design where the members are deforming because of high speed motion and large external forces. Computational times to perform a single analysis are enormous for problems of this type involving large deformations with nonlinear material characteristics. Thus the goal of the methods developed was to reduce the number of analyses, even at the expense of increased computational effort in the optimization technique itself, i.e. additional effort in finding new candidate design points to analyze.

The methods developed take advantage of the special characteristics of the optimization problem, similar to the optimality criterion techniques. This greatly improves their efficiency. For most mechanism design problems, the objective function can be formulated as a sum of squared quantities such as the difference between the desired performance and the actual performance of the mechanism at specified points during its motion. Thus the techniques were developed specifically to handle problems of this type, although the methods are applicable to objective functions which are general sums of nonnegative quantities, such as weight. Many mechanism problems have constraints which are only linear functions of the design variables. Thus a special method was developed for problems of this type. Other problems have constraints which are linear or quadratic, and another method was developed for this case. Some mechanism design problems have more general nonlinear constraints. Methods to handle these cases are currently being developed.

All of the techniques developed in this work have been based on Gauss' method [1] which is applicable to problems without constraints. Wilde [2] has shown this method to be particularly efficient on simple mechanism design problems. The research presented in this paper has extended this method to handle various types of constraints common to more complex mechanism design problems.

#### FORMULATION

For an unconstrained sum-of-squares objective function

$$f(\hat{x}) = \hat{\Phi}^T \hat{\Phi}, \quad (1)$$

where  $\hat{\Phi}$  is a vector of linear or nonlinear functions  $\phi_1$  thru  $\phi_p$  in  $\hat{x}$ , the Gauss method for calculating the next iteration of the design variables,  $\hat{x}_{k+1}$ , given a current design,  $\hat{x}_k$ , is:

$$\hat{x}_{k+1} = \hat{x}_k - \left[ J(\hat{x}_k) J^T(\hat{x}_k) \right]^{-1} J(\hat{x}_k) \hat{\Phi}(\hat{x}_k), \quad (2)$$

where

$$J(\hat{x}) = \begin{bmatrix} \frac{\partial \phi_1}{\partial \hat{x}} & \dots & \frac{\partial \phi_p}{\partial \hat{x}} \end{bmatrix} = \nabla \hat{\Phi}^T. \quad (3)$$

It is observed that only first derivatives of the  $\phi$  functions are required and that the new design point is calculated directly from the current design without using a step length determination with associated one dimensional minimization. Himmelblau [1] has shown this method to be very efficient for unconstrained minimization problems.

This technique has been extended to handle linear inequality constraints of the form

$$g_i(\hat{x}) = \hat{b}_i^T \hat{x} - c_i \leq 0, \quad i = 1, \dots, M \quad (4)$$

as well as equality constraints

$$h_i(\hat{x}) = \hat{d}_i^T \hat{x} - e_i = 0, \quad i = 1, \dots, L. \quad (5)$$

In the derivation of the optimization method, the  $\phi_i$  functions are assumed to be linear in  $x$  of the form

$$\hat{\phi} = J^T \hat{x} + \hat{u}. \quad (6)$$

where  $J$  is a constant matrix. However the resulting technique is applicable to problems in which the  $\phi_i$ 's are general nonlinear functions of  $x$ .

At iteration  $k$ , the  $L$  equality constraints and any of the inequality constraints that are active can be combined and written in the form

$$B^T \hat{x} - \hat{C} = 0. \quad (7)$$

If at the next iteration,  $k+1$ , the variables  $\hat{x}_{k+1}$  are at the optimum design, then the Kuhn-Tucker conditions will be satisfied

$$\nabla f(\hat{x}_{k+1}) + B\hat{\lambda} = 0 \quad (8)$$

$$B^T \hat{x}_{k+1} - \hat{C} = 0 \quad (9)$$

and

$$\hat{\lambda} \geq 0 \quad (10)$$

where  $\hat{\lambda}$  is the vector of Lagrange multipliers. The gradient of  $f$  is given by

$$\nabla f(\hat{x}) = 2J\hat{\phi}(\hat{x}). \quad (11)$$

Expanding  $\hat{\phi}(\hat{x})$  in a Taylor series results in

$$\hat{\phi}(\hat{x}_{k+1}) = \hat{\phi}(\hat{x}_k) + J^T [\hat{x}_{k+1} - \hat{x}_k] + (\text{higher order terms}) \quad (12)$$

It is noted that the higher order terms are equal to zero if  $\hat{\phi}$  is linear. If  $\hat{\phi}$  is not linear then these terms will be neglected and the expansion is only approximate. If equation (12) is substituted into equation (11), evaluated at  $\hat{x}_{k+1}$ , the result is

$$\nabla f(\hat{x}_{k+1}) = 2J \left[ \hat{\phi}(\hat{x}_k) + J^T [\hat{x}_{k+1} - \hat{x}_k] \right]. \quad (13)$$

This may be substituted into the first Kuhn-Tucker condition,



equation (8), and then solved for  $\hat{x}_{k+1}$ :

$$\hat{x}_{k+1} = \hat{x}_k - [2JJ^T]^{-1} [2J\hat{\phi}(\hat{x}_k) + B\hat{\lambda}]. \quad (14)$$

Plugging this equation for  $\hat{x}_{k+1}$  into the second Kuhn-Tucker condition, equation (9), yields

$$B^T \hat{x}_k - \hat{C} - B^T [2JJ^T]^{-1} [2J\hat{\phi}(\hat{x}_k) + B\hat{\lambda}] = 0. \quad (15)$$

If the same set of constraints that are active at  $\hat{x}_{k+1}$  were also active at  $\hat{x}_k$ , then  $B^T \hat{x}_k - \hat{C} = 0$ . Using this result, equation (15) can be solved for  $\hat{\lambda}$  as

$$\hat{\lambda} = - \left[ B [2JJ^T]^{-1} B \right]^{-1} B^T [2JJ^T]^{-1} 2J\hat{\phi}(\hat{x}_k). \quad (16)$$

Substituting this back into equation (14) and simplifying yields an iterative expression for  $\hat{x}_{k+1}$  which will give the optimum solution if the constraints that are active at the optimum point (iteration  $k+1$ ) are active at iteration  $k$ :

$$\hat{x}_{k+1} = \hat{x}_k - \left[ I - [JJ^T]^{-1} B [B^T [JJ^T]^{-1} B]^{-1} B^T \right] [JJ^T]^{-1} J\hat{\phi}(\hat{x}_k). \quad (17)$$

This expression is equivalent to that derived by Paradis and Willmert [3] using a Gradient Projection method as the foundation. The technique converges to the optimal design in one iteration if the objective function,  $f$ , is quadratic and the starting point is on the constraints which are active at the optimal design. If  $f$  is not quadratic, the technique can still be applied, but it will generally require several iterations to reach the optimal design. When the technique terminates, the Kuhn-Tucker conditions will be satisfied independent of the form of the objective function.

Paradis and Willmert demonstrated the efficiency of this method by solving several examples. One example presented was the optimal design of a four-bar mechanism to generate a desired coupler point path. The Gauss constrained technique was compared with the Davidon-Fletcher-Powell method using an interior penalty function approach to handle constraints. Using four different starting points, the Gauss constrained method required from 23 to 33 objective function evaluations whereas the Davidon-Fletcher-Powell method required from 209 to 622. While not all starting points yielded the same optimal design, both methods reached the same local minimum from each starting point. Other examples also showed considerable improvement over existing methods.

The Gauss method has also been extended to include quadratic inequality constraints or quadratic approximations to higher order nonlinear constraints. In this work the constraints are assumed to

have the form

$$q_i(\hat{x}) = \frac{1}{2} \hat{x}^T A_i \hat{x} + \hat{B}_i^T \hat{x} - C_i \leq 0, \quad i = 1, \dots, M. \quad (18)$$

If at iteration  $k+1$  there are  $r$  active constraints ( $r \leq M$ ), the Kuhn-Tucker conditions will be

$$\nabla f(\hat{x}_{k+1}) + \sum_{j=1}^r [A_j \hat{x}_{k+1} + \hat{B}_j] \lambda_j = 0 \quad (19)$$

$$\frac{1}{2} \hat{x}_{k+1}^T A_j \hat{x}_{k+1} + \hat{B}_j^T \hat{x}_{k+1} - C_j = 0, \quad j = 1, \dots, r \quad (20)$$

and

$$\hat{\lambda} \geq 0 \quad (21)$$

where the summation in equation (19) and the  $j$  subscript in equation (20) refer to the set of active constraints only.

Using a derivation similar to that for linear constraints, substituting the expression for the gradient of  $f$ , equation (13), into the first Kuhn-Tucker condition, equation (19), and solving for  $\hat{x}_{k+1}$  produces

$$\hat{x}_{k+1} = \left[ 2JJ^T + \sum_{j=1}^r A_j \lambda_j \right]^{-1} \left[ 2JJ^T \hat{x}_k - \nabla f(\hat{x}_k) - \sum_{j=1}^r \hat{B}_j \lambda_j \right]. \quad (22)$$

This expression for  $\hat{x}_{k+1}$  in terms of  $\hat{\lambda}$  is now substituted into the second Kuhn-Tucker condition, equation (20), to obtain:

$$\begin{aligned} & \frac{1}{2} \left[ \left[ 2JJ^T + \sum_{j=1}^r A_j \lambda_j \right]^{-1} \left[ 2JJ^T \hat{x}_k - \nabla f(\hat{x}_k) - \sum_{j=1}^r \hat{B}_j \lambda_j \right] \right]^T \\ & A_i \left[ \left[ 2JJ^T + \sum_{j=1}^r A_j \lambda_j \right]^{-1} \left[ 2JJ^T \hat{x}_k - \nabla f(\hat{x}_k) - \sum_{j=1}^r \hat{B}_j \lambda_j \right] \right] \\ & + B_i \left[ \left[ 2JJ^T + \sum_{j=1}^r A_j \lambda_j \right]^{-1} \left[ 2JJ^T \hat{x}_k - \nabla f(\hat{x}_k) - \sum_{j=1}^r \hat{B}_j \lambda_j \right] \right] \\ & - C_i = 0, \quad i = 1, \dots, r. \end{aligned} \quad (23)$$

These  $r$  nonlinear equations in terms of the new unknowns,  $\lambda_1$  thru  $\lambda_r$ , and the old design variables,  $\hat{x}_k$ , are solved by an iterative process for the values of  $\lambda_1$  thru  $\lambda_r$ . The lambda values are then substituted into equation (22) which will yield new values for the design

variables. It is observed that the matrix  $2JJ^T$  is the matrix of second partial derivatives,  $G$ , of the objective function if it is quadratic. Thus, by replacing  $2JJ^T$  in equation (22) and (23), this technique becomes a modification of the second order method rather than the Gauss method.

At the optimum design all constraints will either be satisfied (less than zero) or active (equal to zero) and each active constraint will have a corresponding lambda whose value is greater than or equal to zero. If, at some iteration the set of design variables yields a violated constraint, then obviously the optimum point has not been reached. In this case, the newly violated constraint will be added to the set of active constraints and the procedure allowed to continue. If at some iteration the set of design variables yields all active or satisfied constraints, but one or more of the active constraints has a corresponding negative lambda, then the optimum design has also not been reached. The negative lambda implies that the iteration process would like to move away from the corresponding constraint boundary toward the feasible region where the constraint is satisfied. Thus, the constraint is dropped from the set of active constraints and the process allowed to continue. If more than one negative lambda existed, then constraints are dropped one at a time starting with the constraint with the most negative lambda.

A constraint is added to the set of active constraints if it should become either active (equal to zero) or violated (greater than zero) when the step is taken from  $x_k$  to  $x_{k+1}$ . In the case where a constraint becomes violated, a line is "drawn" between  $x_k$  and  $x_{k+1}$  and the actual step is taken to the farthest point along the line so that no constraints are violated. In effect, this procedure is the same as stepping back from  $x_{k+1}$  toward  $x_k$  until the newly violated constraint is just active (equal to zero). The constraint is then added to the set of active constraints for the next iteration.

An example problem with quadratic constraints given by Boston, Willmert and Sathyamoorthy [4] shows this method to be very efficient when compared to the generalized reduced gradient method (GRG). The problem consisted of finding the optimal design of a four-bar mechanism (minimizing the coupler point path error with respect to a given path) subject to several linear constraints on link length and movability. Additionally, constraints were placed on the crank pin to limit its location to the intersection of two circular (quadratic) regions. The program was run for a four by four matrix of problems which included four different starting points and four different conditions of the quadratic constraints. For all sixteen runs, the number of objective function evaluations for this new method ranged from 8 to 32 (average was 15), while the GRG method required from 303 to 699 (average was 502) evaluations.

The interesting information here is that the solution to this quadratically constrained four-bar mechanism used no more objective function evaluations than the linearly constrained four-bar mechanism example by Paradis and Willmert. While these two examples are necessarily different, this tendency toward requiring similar numbers

of function evaluations for different classes of problems is very desirable. The net result is that we now have an optimization procedure for sum of squared quantities objective functions subject to linear and quadratic constraints that not only requires relatively few function evaluations, but seems to be constraint order independent. Now the need is to determine a method which will also work for higher order constraints.

Boston, et al. [4] attempted to apply this method to higher order problems, but met with mixed results. The problems encountered seemed to tie in with the higher order constraints rather than with the higher order objective functions. There are several limitations implicit in the algorithm which appear to be the source of the problems encountered. The first limitation has to do with the application of the constraints, equation (18), to the first Kuhn-Tucker condition, equation (19). When approximating a higher order function by a quadratic Taylor series expansion about some point  $\hat{x}_0$ , not only is the  $A_i$  matrix a function of  $\hat{x}_0$ , but so is the  $B_i$  vector and the  $C_i$  scalar. Thus the constraint approximation, equation (18), should be written as

$$g_i(\hat{x}) = \frac{1}{2} \hat{x}^T A_i(\hat{x}_0) \hat{x} + \left[ \hat{B}_i(\hat{x}_0) \right]^T \hat{x} - C_i(\hat{x}_0) \leq 0, \quad i = 1, \dots, M \quad (24)$$

where

$$\hat{x} = \hat{x}_0 + \Delta \hat{x}. \quad (25)$$

As  $\hat{x}$  approaches  $\hat{x}_0$  (or  $\Delta \hat{x}$  approaches 0) then this approximation approaches the exact value of the constraint. Thus as the algorithm progresses along and constraints are added and dropped, the constraints must be reapproximated at the latest design to keep the step size small. This can be achieved by taking the new values of  $\hat{x}$  as generated by equation (22) and substituting them into the actual constraint equations to get improved values for the  $A_i(\hat{x}_0)$ ,  $\hat{B}_i(\hat{x}_0)$ , and  $C_i(\hat{x}_0)$  terms in equation (24) with respect to the current design point.

The second limitation involves the second Kuhn-Tucker condition, equation (20), which is used to obtain the equation for the new values of  $\lambda$ , equation (23). This is simply the equation for the active constraints. In the original formulation, an attempt was made at obtaining a linear approximation in  $\lambda$  for this constraint equation. This would allow equation (23) to be solved explicitly for  $\lambda$ . However, failing this an iterative procedure was employed to find the values for  $\lambda$ . Now that an iterative process is required there is no advantage in keeping a quadratic approximation when the actual constraint will work just as well. Replacing equation (20) with the active nonlinear constraint equations will remove any errors due to the approximation process.

The stepping back procedure for violated constraints, described above, can also be a source of problems. With non-convex programming problems this procedure may lead to a situation where the algorithm cannot move away from a non-optimum design. Because the stepping

back procedure assumed a straight line path between the two design points, it is possible, when backing out of a newly violated constraint, to move into the violated region of the constraint that was active at the beginning of the step. The procedure would then step back still further until all constraints are satisfied. It is possible to end up with the same set of active constraints as at the start of the iteration. In this case the next iteration will produce the same design, which may be non-optimal.

Two alternatives are readily apparent which may solve this problem. The first one is that when a constraint becomes violated, repeat the step but include the newly violated constraint in the set of active constraints. The second alternative is to move to the point where the constraint is violated, and then iterate from there without stepping back. Of course, the violated constraint is added to the set of active constraints. Boston, et.al. [4] looked into this second alternative to some extent. They reported that it did not always work. However, it is not clear if it was the "no stepping back" that was the cause of the problems or if the second order approximations to the constraints contributed to the problems.

In summary, the Gauss nonlinearly constrained technique is very effective at solving quadratically constrained problems. No major difficulties appear to exist which would preclude it from solving problems with higher order constraints once the modifications discussed above are implemented. This method with the proposed modifications is currently the leading candidate as the best method for solving highly nonlinear mechanism design problems.

## RESULTS

A verification of the effectiveness of the Gauss constrained method applied to problems with quadratic constraints is obtained by solving the Rosen-Suzuki test problem [5]:

$$\text{minimize } F(\hat{x}) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4$$

subject to:

$$g_1(\hat{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8 \leq 0$$

$$g_2(\hat{x}) = x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10 \leq 0$$

$$g_3(\hat{x}) = 2x_1^2 + x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_2 - x_4 - 5 \leq 0$$

The optimum design for this problem is at  $\hat{x} = [0, 1, 2, -1]$

Two versions of the Gauss nonlinearly constrained technique and the generalized reduced gradient method, identified as GRG, were used

from four different starting points. One version of the Gauss nonlinearly constrained technique, identified as GNLC, uses the stepping back procedure and requires a feasible starting design and will always maintain a feasible design. The other version, identified as GNLC.NS, does not use the stepping back procedure and has no requirement on the feasibility of the design at any stage of the optimization. The results are summarized in Table I. It can easily be seen that the Gauss nonlinearly constrained technique is much more efficient with respect to number of function evaluations than the generalized reduced gradient method.

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ALGORITHM	STARTING DESIGN, $x_0$	NUMBER OF ITERATIONS	FUNCTION EVALUATIONS
GNLC.NS	[0,0,0,0]	2	3
GC.NS	[1,1,1,1]	2	3
GC.NS	[2,2,2,2]	2	3
GNLC.NS	[0,0, $\sqrt{5}$ ,0]	3	4
GNLC	[0,0, $\sqrt{5}$ ,0]	3	4
GNLC	[0,0,0,0]	5	6
GNLC	[1,1,1,1]	5	6
GRG	[0,0, $\sqrt{5}$ ,0]	9	83
GRG	[0,0,0,0]	11	106
GRG	[1,1,1,1]	11	133
GRG	[2,2,2,2]	11	144

Table I: Comparison of Algorithms

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#### CONCLUSIONS

The optimization techniques developed in this research as extensions of the Gauss method to handle various types of constraints are effective approaches to reducing the number of analyses required to obtain an optimal design. As a result, the computational time for large problems should be reduced significantly.

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